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Effect of viscous dissipation on mixed convection in a vertical channel with boundary conditions of the third kind

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Abstract

The effect of viscous dissipation on fully-developed mixed convection is analysed for the laminar flow in a parallel-plate vertical channel whose walls exchange heat with an external fluid. Both conditions of equal and of different reference temperatures of the external fluid are considered. First, the simpler cases of either negligible Brinkman number or negligible Grashof number are solved analytically. Then, the combined effects of buoyancy forces and of viscous dissipation are analysed by a perturbation series method. In the examined cases, the velocity field, the temperature field and the Nusselt numbers are evaluated.

Nomenclature

A constant defined by equation (13) [Pa m^{-1}]
a_n, b_n, c_n dimensionless coefficients defined by equation (64)
Bi_1, Bi_2 Biot numbers defined in equation (19)
Br Brinkman number defined in equation (19)
c_p specific heat at constant pressure [J kg^{-1} K^{-1}]
d_n dimensionless coefficients defined by equation (66)
D $=2L$, hydraulic diameter [m]
F dimensionless parameter defined by equation (50)
$\mathbf{g}$ acceleration due to gravity [m s^{-2}]
Gr Grashof number defined in equation (19)
h_1, h_2 external heat transfer coefficients [W m^{-2} K^{-1}]
j non-negative integer number
k thermal conductivity [W m^{-1} K^{-1}]
L channel width [m]
n non-negative integer number
Nu_1, Nu_2 Nusselt numbers defined by equation (31)
p pressure [Pa]
P $=p+\rho_0 \mathbf{g} X$, difference between the pressure and the hydrostatic pressure [Pa]
Pr Prandtl number defined in equation (19)
Re Reynolds number defined in equation (19)
R_T temperature difference ratio defined in equation (19)
S dimensionless parameter defined in equation (19)
T temperature [K]
T_1, T_2 reference temperatures of the external fluid [K]
T_0 reference temperature defined in equation (20)
u dimensionless velocity in the X-direction, defined in equation (19)
$u_n(y)$ dimensionless functions defined by equation (52)
$\bar{u}$ mean value of u defined by equation (23)
U velocity component in the X-direction [m s^{-1}]
U_0 reference velocity defined in equation (20)
$\mathbf{U}$ velocity [m s^{-1}]
X streamwise coordinate [m]
y dimensionless transverse coordinate defined in equation (19)
Y transverse coordinate [m].

Greek symbols
α $=k/(\rho_0 c_p)$, thermal diffusivity [m^2 s^{-1}]
β thermal expansion coefficient [K^{-1}]
ΔT reference temperature difference defined either by equation (21) or by equation (22)
ε dimensionless parameter defined by equation (51)
θ dimensionless temperature defined in equation (19)
θ_b dimensionless bulk temperature defined by equation (24)
μ dynamic viscosity [Pa s]
ν $=\mu/\rho_0$, kinematic viscosity [m^2 s^{-1}]
Ξ dimensionless parameter defined in equation (19)

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Ξ_c critical value of Ξ for the onset of flow reversal
ρ mass density [kg m^{-3}]
ρ_0 value of the mass density when $T = T_0$ [kg m^{-3}].

1. Introduction

Several papers on mixed convection in a parallel-plate vertical channel are available in the literature. However, most of these studies are based on the hypothesis that the effect of viscous dissipation is negligible. The fully-developed region has been studied analytically. For instance, the boundary condition of uniform wall temperatures has been analysed by Aung and Worku [1]. The cases of either uniform temperature or uniform heat flux at each boundary surface have been studied by Cheng, Kou and Huang [2] and by Hamadah and Wirtz [3]. The boundary condition of linearly varying wall temperatures has been considered by Tao [4]. Some studies on the developing flow have been carried out by numerical methods. Aung and Worku [5, 6] have studied the developing flow with asymmetric wall temperatures [5] and with asymmetric wall heat fluxes [6]. The developing flow with asymmetric wall temperatures has been considered also by Ingham, Keen and Heggs [7], with particular reference to situations where reverse flows occur. All the studies quoted above, as well as the references quoted in the review of the literature on this subject presented by Aung [8], assume that viscous dissipation effects are negligible. On the other hand, Barletta [9] and Zanchini [10] have pointed out that relevant effects of viscous dissipation on the temperature profiles and on the Nusselt numbers may occur in the fully-developed laminar forced convection in tubes. Thus, an analysis of the effects of viscous dissipation in the fully-developed mixed convection in vertical ducts appears as interesting.

Recently, the laminar mixed convection with viscous dissipation in the fully-developed region of a parallel-plate vertical channel has been studied by Barletta [11]. The author has analysed the boundary condition of uniform wall temperatures, and has considered both different and equal wall temperatures. The aim of this paper is to extend the analysis performed in ref. [11], by assuming that the walls of the channel exchange heat with an external fluid. Both equal and different reference temperatures of the external fluid, as well as both different and equal Biot numbers, are considered. In the limit of infinite Biot numbers at both walls, the solutions obtained in this paper coincide with those presented in ref. [11].

2. Mathematical model

In this section, the momentum balance equation, the energy balance equation and the boundary conditions are settled in a form suitable for the solution of the problem under examination. Then, these equations are rewritten in dimensionless form.

Let us consider the steady and laminar flow of a Newtonian fluid in the fully-developed region of a parallel-plate vertical channel. The X-axis lies on the axial plane of the channel, and its direction is opposite to the gravitational field. The Y-axis is orthogonal to the walls. The channel occupies the region of space $-L/2 \leqslant Y \leqslant L/2$. The thermal conductivity, the thermal diffusivity, the dynamic viscosity and the thermal expansion coefficient of the fluid will be assumed to be constant. As customary, the Boussinesq approximation and the equation of state

$$\rho = \rho_0[1-\beta(T-T_0)] \tag{1}$$

will be adopted. Moreover, it will be assumed that the only nonzero component of the velocity field $\mathbf{U}$ is the X-component U. Thus, since $\nabla \cdot \mathbf{U} = 0$, one has

$$\frac{\partial U}{\partial X} = 0 \tag{2}$$

so that U depends only on Y. The momentum balance equations along X and Y yield [12]

$$\beta \mathbf{g}(T-T_0) - \frac{1}{\rho_0}\frac{\partial P}{\partial X} + \nu \frac{\mathrm{d}^2 U}{\mathrm{d}Y^2} = 0 \tag{3}$$

$$\frac{\partial P}{\partial Y} = 0 \tag{4}$$

where $P = p + \rho_0 \mathbf{g} X$. Since, on account of equation (4), P depends only on X, equation (3) can be rewritten as

$$T - T_0 = \frac{1}{\beta \mathbf{g} \rho_0}\frac{\mathrm{d}P}{\mathrm{d}X} - \frac{\nu}{\beta \mathbf{g}}\frac{\mathrm{d}^2 U}{\mathrm{d}Y^2}. \tag{5}$$

From equation (5), one obtains

$$\frac{\partial T}{\partial X} = \frac{1}{\beta \mathbf{g} \rho_0}\frac{\mathrm{d}^2 P}{\mathrm{d}X^2} \tag{6}$$

$$\frac{\partial T}{\partial Y} = -\frac{\nu}{\beta \mathbf{g}}\frac{\mathrm{d}^3 U}{\mathrm{d}Y^3} \tag{7}$$

$$\frac{\partial^2 T}{\partial Y^2} = -\frac{\nu}{\beta \mathbf{g}}\frac{\mathrm{d}^4 U}{\mathrm{d}Y^4}. \tag{8}$$

Both the walls of the channel will be assumed to have a negligible thickness and to exchange heat by convection with an external fluid. In particular, at $Y = -L/2$ the external convection coefficient will be considered as uniform with the value h_1 and the fluid in the region $Y < -L/2$ will be assumed to have a uniform reference temperature T_1. At $Y = L/2$ the external convection coefficient will be considered as uniform with the value h_2 and the fluid in the region $Y > L/2$ will be supposed to have a uniform reference temperature $T_2 \geqslant T_1$. Therefore, the boundary conditions on the temperature field can be expressed as

$$-k\frac{\partial T}{\partial Y}\bigg|_{Y=-L/2} = h_1[T_1 - T(X, -L/2)] \tag{9}$$

$$-k\frac{\partial T}{\partial Y}\bigg|_{Y=L/2} = h_2[T(X, L/2) - T_2]. \tag{10}$$

On account of equation (7), equations (9) and (10) can be rewritten as

$$\frac{\mathrm{d}^3 U}{\mathrm{d}Y^3}\bigg|_{Y=-L/2} = \frac{\beta g h_1}{k\nu}[T_1 - T(X, -L/2)] \tag{11}$$

$$\frac{\mathrm{d}^3 U}{\mathrm{d}Y^3}\bigg|_{Y=L/2} = \frac{\beta g h_2}{k\nu}[T(X, L/2) - T_2]. \tag{12}$$

It is easily verified that equations (11) and (12) imply that $\partial T/\partial X$ is zero both at $Y = -L/2$ and at $Y = L/2$. Since equation (6) ensures that $\partial T/\partial X$ does not depend on Y, one is led to the conclusion that $\partial T/\partial X$ is zero everywhere. Therefore, the temperature T depends only on Y, i.e. $T = T(Y)$. Thus, on account of equation (6), there exists a constant A such that

$$\frac{\mathrm{d}P}{\mathrm{d}X} = A. \tag{13}$$

For the problem under exam, the energy balance equation in the presence of viscous dissipation can be written as [12]

$$\frac{\mathrm{d}^2 T}{\mathrm{d}Y^2} = -\frac{\nu}{\alpha c_p}\left(\frac{\mathrm{d}U}{\mathrm{d}Y}\right)^2. \tag{14}$$

Equations (8) and (14) yield a differential equation for U, namely

$$\frac{\mathrm{d}^4 U}{\mathrm{d}Y^4} = \frac{\beta g}{\alpha c_p}\left(\frac{\mathrm{d}U}{\mathrm{d}Y}\right)^2. \tag{15}$$

The boundary conditions on U are

$$U(-L/2) = U(L/2) = 0 \tag{16}$$

together with equations (11) and (12), which, on account of equation (5), can be rewritten as

$$\frac{\mathrm{d}^3 U}{\mathrm{d}Y^3}\bigg|_{Y=-L/2} - \frac{h_1}{k}\frac{\mathrm{d}^2 U}{\mathrm{d}Y^2}\bigg|_{Y=-L/2} = -\frac{Ah_1}{k\mu} - \frac{\beta g h_1}{k\nu}(T_0 - T_1) \tag{17}$$

$$\frac{\mathrm{d}^3 U}{\mathrm{d}Y^3}\bigg|_{Y=L/2} + \frac{h_2}{k}\frac{\mathrm{d}^2 U}{\mathrm{d}Y^2}\bigg|_{Y=L/2} = \frac{Ah_2}{k\mu} - \frac{\beta g h_2}{k\nu}(T_2 - T_0). \tag{18}$$

Equations (15)–(18) determine the velocity distribution. They can be written in a dimensionless form by means of the following dimensionless parameters:

$$u = \frac{U}{U_0}, \quad \theta = \frac{T - T_0}{\Delta T}, \quad y = \frac{Y}{D},$$

$$Gr = \frac{g\beta\Delta T D^3}{\nu^2}, \quad Re = \frac{U_0 D}{\nu}, \quad Br = \frac{\mu U_0^2}{k\Delta T},$$

$$Pr = \frac{\nu}{\alpha}, \quad \Xi = \frac{Gr}{Re}, \quad R_T = \frac{T_2 - T_1}{\Delta T}$$

$$Bi_1 = \frac{h_1 D}{k}, \quad Bi_2 = \frac{h_2 D}{k}, \quad S = \frac{Bi_1\,Bi_2}{Bi_1\,Bi_2 + 2Bi_1 + 2Bi_2}. \tag{19}$$

In equation (19), $D = 2L$ is the hydraulic diameter, while the reference velocity U_0 and the reference temperature T_0 are given by

$$U_0 = -\frac{AD^2}{48\mu}$$

$$T_0 = \frac{T_1 + T_2}{2} + S\left(\frac{1}{Bi_1} - \frac{1}{Bi_2}\right)(T_2 - T_1). \tag{20}$$

The reference temperature difference ΔT is given either by

$$\Delta T = T_2 - T_1 \tag{21}$$

if $T_1 < T_2$, or by

$$\Delta T = \frac{\nu^2}{c_p D^2} \tag{22}$$

if $T_1 = T_2$. Therefore, as in Ref. [11], the value of the dimensionless parameter R_T can be either 0 or 1. More precisely, R_T equals 1 for asymmetric fluid temperatures, $T_1 < T_2$, and equals 0 for symmetric fluid temperatures, $T_1 = T_2$.

The dimensionless mean velocity $\bar{u}$ and the dimensionless bulk temperature θ_b are given by [11]

$$\bar{u} = 2\int_{-1/4}^{1/4} u\,\mathrm{d}y \tag{23}$$

$$\theta_b = \frac{2}{\bar{u}}\int_{-1/4}^{1/4} u\theta\,\mathrm{d}y. \tag{24}$$

On account of equation (13), for upward flow $A < 0$, so that U_0, Re and Ξ are positive. For downward flow $A > 0$, while U_0, Re and Ξ are negative. By employing the dimensionless quantities defined in equation (19), equations (15)–(18) can be rewritten as

$$\frac{\mathrm{d}^4 u}{\mathrm{d}y^4} = \Xi\,Br\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 \tag{25}$$

$$u(-1/4) = u(1/4) = 0 \tag{26}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2}\bigg|_{y=-1/4} - \frac{1}{Bi_1}\frac{\mathrm{d}^3 u}{\mathrm{d}y^3}\bigg|_{y=-1/4} = -48 + \frac{R_T\Xi}{2}S\left(4 + \frac{1}{Bi_1}\right) \tag{27}$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2}\bigg|_{y=1/4} + \frac{1}{Bi_2}\frac{\mathrm{d}^3 u}{\mathrm{d}y^3}\bigg|_{y=1/4} = -48 - \frac{R_T\Xi}{2}S\left(4 + \frac{1}{Bi_2}\right). \tag{28}$$

Similarly, equations (14) and (19) yield

$$\frac{d^2\theta}{dy^2}+Br\left(\frac{du}{dy}\right)^2=0 \tag{29}$$

while from equations (5) and (19) one obtains

$$\theta=-\frac{1}{\Xi}\left(48+\frac{d^2u}{dy^2}\right). \tag{30}$$

Equations (25)–(30) show that the dimensionless velocity profile and the dimensionless temperature profile depend on five parameters: the ratio $\Xi = Gr/Re$, the Brinkman number Br, the temperature difference ratio R_T and the Biot numbers Bi_1 and Bi_2. A Nusselt number can be defined at each boundary, as follows:

$$Nu_1=\frac{D}{R_T[T(L/2)-T(-L/2)]+(1-R_T)\Delta T}\left.\frac{dT}{dY}\right|_{Y=-L/2}$$
$$Nu_2=\frac{D}{R_T[T(L/2)-T(-L/2)]+(1-R_T)\Delta T}\left.\frac{dT}{dY}\right|_{Y=L/2}. \tag{31}$$

The Nusselt numbers Nu_1 and Nu_2 can be employed to evaluate the heat fluxes at the walls. In fact, the heat flux per unit area is given by $q_1 = -k(dT/dY)|_{-L/2}$ at the left wall, and by $q_2 = -k(dT/dY)|_{L/2}$ at the right wall. Let us first assume $R_T = 1$. Then, from equation (31) one obtains

$$q_1=-\frac{k\,Nu}{D}[T(L/2)-T(-L/2)]$$
$$q_2=-\frac{k\,Nu_2}{D}[T(L/2)-T(-L/2)]. \tag{32}$$

The heat flux densities q_1 and q_2 can be also expressed as

$$q_1=-h_1[T(-L/2)-T_1]$$
$$q_2=-h_2[T_2-T(L/2)]. \tag{33}$$

Equations (32) and (33) yield

$$T(L/2)-T(-L/2)=\frac{T_2-T_1}{1+\frac{k}{D}\left(\frac{Nu_1}{h_1}+\frac{Nu_2}{h_2}\right)}. \tag{34}$$

Let us now assume $R_T = 0$. Equation (31) yields

$$q_1=-\frac{k\,Nu_1}{D}\Delta T$$
$$q_2=-\frac{k\,Nu_2}{D}\Delta T \tag{35}$$

where ΔT is the reference temperature difference defined by equation (22). By employing equation (19), equation (31) can be written as

$$Nu_1=\frac{1}{R_T[\theta(1/4)-\theta(-1/4)]+(1-R_T)}\left.\frac{d\theta}{dy}\right|_{y=-1/4}$$
$$Nu_2=\frac{1}{R_T[\theta(1/4)-\theta(-1/4)]+(1-R_T)}\left.\frac{d\theta}{dy}\right|_{y=1/4}. \tag{36}$$

3. Separated effects of buoyancy forces and viscous dissipation

In this section, the simpler cases of either negligible viscous dissipation or negligible buoyancy forces will be solved analytically. As far as the author knows, these solutions are not yet available in the literature, for boundary conditions of the third kind.

The case of negligible viscous dissipation can be obtained by setting $Br = 0$ in the dimensionless energy equation (29). As a consequence, the dimensionless temperature field θ is independent of the dimensionless velocity field u. Moreover, equations (25)–(28) can be easily solved and yield

$$u=\left(24+\frac{S\Xi R_T y}{3}\right)\left(\frac{1}{16}-y^2\right). \tag{37}$$

With $Bi_1 = Bi_2 = Bi$, equation (37) can be rewritten as

$$u=\left[24+\frac{Bi\,\Xi R_T y}{3(Bi+4)}\right]\left(\frac{1}{16}-y^2\right). \tag{38}$$

In the limit $Bi \to +\infty$, one obtains the special case in which the boundaries of the channel are kept at the temperatures T_1 and T_2, respectively. In this limit, equation (19) yields $S \to 1$, so that equation (37) reduces to

$$u=\left(24+\frac{\Xi R_T y}{3}\right)\left(\frac{1}{16}-y^2\right) \tag{39}$$

i.e. to the velocity profile determined by Barletta [11].

By substituting equation (37) in equations (23) and (24), one obtains

$$\bar{u}=1,\quad \theta_b=\frac{S^2\Xi R_T}{2880}. \tag{40}$$

Moreover, equations (30), (36) and (37) yield

$$\theta=2SR_T y,\quad Nu_1=Nu_2=2R_T. \tag{41}$$

Equation (41) points out the following results, in the limit $Br = 0$. For asymmetric fluid temperatures, i.e. for $R_T = 1$, heat is transferred by pure conduction; for symmetric fluid temperatures, i.e. for $R_T = 0$, the temperature is uniform and no heat transfer occurs.

For symmetric fluid temperatures, for asymmetric fluid temperatures with either $Bi_1 \to 0$ or $Bi_2 \to 0$ and for asymmetric fluid temperatures with $\Xi = 0$, equation (37) yields the usual Hagen–Poiseuille velocity profile. Indeed, as is shown by equation (41), both for symmetric fluid temperatures and for asymmetric fluid temperatures with either $Bi_1 \to 0$ or $Bi_2 \to 0$, the temperature is uniform, so that no buoyancy force can be present. Finally, the hypothesis $\Xi = 0$ implies $Gr = 0$, i.e. vanishing buoyancy forces.

As is shown by equation (37), in the case of asymmetric fluid temperatures with dominant buoyancy forces, i.e. $\Xi \to \pm\infty$, the dimensionless velocity u defined by equation (19) diverges. Then, it is convenient to define as dimensionless velocity the quantity

$$\frac{u}{\Xi} = \frac{UD}{\nu\, Gr} \tag{42}$$

which is independent of U_0. For $\Xi \to \pm\infty$, equation (37) yields the dimensionless velocity profile for free convection

$$\frac{u}{\Xi} = \frac{Sy}{3}\left(\frac{1}{16} - y^2\right). \tag{43}$$

Let us now analyse the conditions under which the buoyancy forces produce a flow reversal. If $T_1 < T_2$ and $U_0 > 0$ (upward flow), equation (37) implies that there exists a positive critical value Ξ_c such that, for $\Xi > \Xi_c$, a flow reversal occurs at $y = -1/4$. The value of Ξ_c, which can be obtained by setting $\mathrm{d}u/\mathrm{d}y = 0$ at $y = -1/4$, is

$$\Xi_c = \frac{288}{S}. \tag{44}$$

If $T_1 < T_2$ and $U_0 < 0$ (downward flow), equation (37) ensures that there exists a negative critical value Ξ_c such that, for $\Xi < \Xi_c$, a flow reversal occurs at $y = 1/4$. This critical value of Ξ, obtained by setting $\mathrm{d}u/\mathrm{d}y = 0$ at $y = 1/4$, is

$$\Xi_c = -\frac{288}{S}. \tag{45}$$

Plots of u vs. y, evaluated through equation (37), are reported in Fig. 1 for $\Xi = 0$, $\Xi = 500$ and $\Xi = 1000$ and for $Bi_1 = Bi_2 = 10$. With this value of the Biot numbers, equations (19) and (45) yield $\Xi_c = 403.2$ Indeed, in Fig. 1, the plots for $\Xi = 500$ and for $\Xi = 1000$ displays a flow reversal next to the boundary $y = -1/4$.

Let us now consider the case of negligible buoyancy forces with a relevant viscous dissipation, which corresponds to $\Xi = Gr/Re = 0$. Since a purely forced convection occurs in this case, the Hagen–Poiseuille velocity profile,

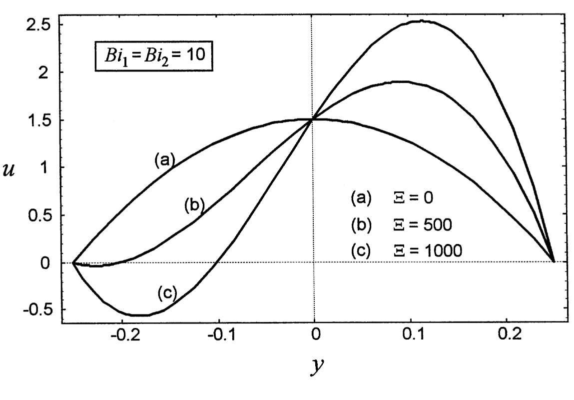

Fig. 1. Plots of u vs. y in the case $R_T = 1$, for some values of Ξ, $Br = 0$ and $Bi_1 = Bi_2 = 10$.

$$u = 24\left(\frac{1}{16} - y^2\right) \tag{46}$$

is present within the channel. Indeed, both for symmetric and for asymmetric fluid temperatures, equation (46) is the solution of equations (25)–(30) when $\Xi = 0$. Equations (9), (10) and (19) yield the boundary conditions on θ, i.e.,

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}y}\right|_{y=-1/4} = Bi_1\left[\frac{SR_T}{2}\left(1 + \frac{4}{Bi_1}\right) + \theta(-1/4)\right]$$

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}y}\right|_{y=1/4} = Bi_2\left[\frac{SR_T}{2}\left(1 + \frac{4}{Bi_2}\right) - \theta(1/4)\right]. \tag{47}$$

On account of equations (29), (46) and (47), the temperature profile can be expressed as

$$\theta = -192 Br\, y^4 + 2S\left[R_T + 12Br\left(\frac{1}{Bi_2} - \frac{1}{Bi_1}\right)\right]y + \frac{3}{4}Br\, S\left[\frac{64}{Bi_1\, Bi_2} + 10\left(\frac{1}{Bi_1} + \frac{1}{Bi_2}\right) + 1\right]. \tag{48}$$

Equations (31) and (48) yield

$$Nu_1 = 2\frac{F + 6Br}{1 + R_T(F-1)}, \quad Nu_2 = 2\frac{F - 6Br}{1 + R_T(F-1)} \tag{49}$$

where

$$F = \theta(1/4) - \theta(-1/4) = S\left[R_T + 12Br\left(\frac{1}{Bi_2} - \frac{1}{Bi_1}\right)\right]. \tag{50}$$

Equations (49) and (50) ensure that, for $R_T = 0$, $Nu_1 > 0$ and $Nu_2 < 0$ for every value of Br. On the other hand, for $R_T = 1$, Nu_1 and Nu_2 may become singular and their sign depends on the values of Br, Bi_1 and Bi_2. In particular, for $Br < Bi_1/[6(Bi_1 + 4)]$ both Nu_1 and Nu_2 are positive, while for $Br \geqslant Bi_1/[6(Bi_1 + 4)]$ $Nu_1 > 0$ and $Nu_2 \leqslant 0$. Moreover, both Nu_1 and Nu_2 become singular when F is zero, i.e. when $Bi_2 > Bi_1$ and $Br = Bi_1\, Bi_2/[12(Bi_2 - Bi_1)]$. Indeed, in this special case, Nu_1 and Nu_2 are singular because $\theta(-1/4) = \theta(1/4)$, as it is shown by equation (50). Note that, when $Bi_2 > Bi_1$ and $Br > Bi_1\, Bi_2/[12(Bi_2 - Bi_1)]$, one obtains $T(-L/2) > T(L/2)$ even if $T_1 < T_2$.

Plots of θ vs. y for $R_T = 1$ ($T_1 < T_2$), evaluated through equation (48), are reported in Figs 2 and 3, for some values of Br. Figure 2 refers to $Bi_1 = Bi_2 = 10$, while Fig. 3 refers to $Bi_1 = 1$ and $Bi_2 = 10$. In the latter case, one obtains $Bi_1\, Bi_2/[12(Bi_2 - Bi_1)] = 5/54 \cong 0.0926$. Indeed, in Fig. 3, the plots for $Br = 0.5$ and for $Br = 1$ are such that $T(-L/2) > T(L/2)$.

4. Combined effects of buoyancy forces and viscous dissipation

In this section, both buoyancy forces and viscous dissipation are considered as non-negligible. First, equations

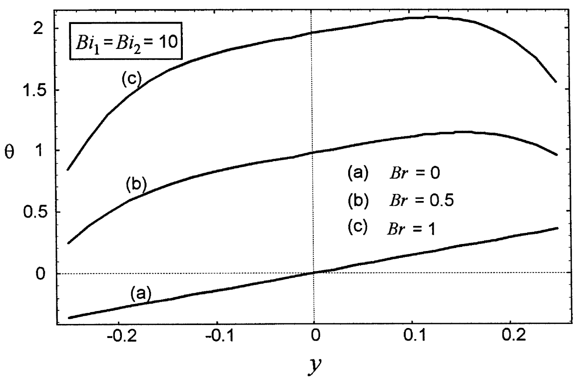

Fig. 2. Plots of θ vs. y in the case $R_T = 1$, for some values of Br, $\Xi = 0$ and $Bi_1 = Bi_2 = 10$.

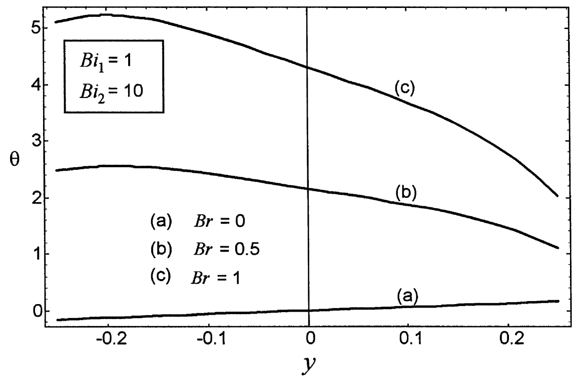

Fig. 3. Plots of θ vs. y in the case $R_T = 1$, for some values of Br, $\Xi = 0$, $Bi_1 = 1$ and $Bi_2 = 10$.

(25)–(28) are solved by a perturbation series method. Then, the dimensionless temperature field is determined by means of equation (30).

As in ref. [11], let us consider the dimensionless parameter

$$\varepsilon = \Xi\, Br = Re\, Pr \frac{\beta g D}{c_p} \tag{51}$$

which is independent of the reference temperature difference ΔT. For fixed values of R_T, Ξ, Bi_1 and Bi_2, the solution of equations (25)–(28) can be expressed by the perturbation expansion

$$u(y) = u_0(y) + u_1(y)\varepsilon + u_2(y)\varepsilon^2 + \cdots = \sum_{n=0}^{\infty} u_n(y)\varepsilon^n. \tag{52}$$

To obtain the solution of equations (25)–(28) with the form (52), one first substitutes equation (52) in equations (25)–(28) and collects terms having like powers of ε. Then, one equates the coefficient of each power of ε to zero [13]. Thus, one obtains a sequence of boundary value problems which can be solved in succession and yield the unknown functions $u_n(y)$.

The boundary value problem for $n = 0$ is

$$\frac{d^4 u_0}{dy^4} = 0 \tag{53}$$

$$u_0(-1/4) = u_0(1/4) = 0 \tag{54}$$

$$\left.\frac{d^2 u_0}{dy^2}\right|_{y=-1/4} - \frac{1}{Bi_1}\left.\frac{d^3 u_0}{dy^3}\right|_{y=-1/4} = -48 + \frac{R_T \Xi}{2} S\left(4 + \frac{1}{Bi_1}\right) \tag{55}$$

$$\left.\frac{d^2 u_0}{dy^2}\right|_{y=1/4} + \frac{1}{Bi_2}\left.\frac{d^3 u_0}{dy^3}\right|_{y=1/4} = -48 - \frac{R_T \Xi}{2} S\left(4 + \frac{1}{Bi_2}\right). \tag{56}$$

The solution of equations (53)–(56) is given by

$$u_0(y) = \left(24 + \frac{S \Xi R_T y}{3}\right)\left(\frac{1}{16} - y^2\right). \tag{57}$$

The right-hand side of equation (57) coincides with that of equation (37) and gives the dimensionless velocity profile in the case $Br = 0$.

The boundary value problem for every integer $n > 0$ is

$$\frac{d^4 u_n}{dy^4} = \sum_{j=0}^{n-1} \frac{du_j}{dy} \frac{du_{n-j-1}}{dy} \tag{58}$$

$$u_n(-1/4) = u_n(1/4) = 0 \tag{59}$$

$$\left.\frac{d^2 u_n}{dy^2}\right|_{y=-1/4} - \frac{1}{Bi_1}\left.\frac{d^3 u_n}{dy^3}\right|_{y=-1/4} = \left.\frac{d^2 u_n}{dy^2}\right|_{y=1/4} + \frac{1}{Bi_2}\left.\frac{d^3 u_n}{dy^3}\right|_{y=1/4} = 0. \tag{60}$$

Since $u_0(y)$ is the known function given by equation (57), an iterative solution of equations (58)–(60) is possible and yields the functions $u_n(y)$, $n > 0$. As a consequence of equations (30), (52) and (57), the dimensionless temperature θ can be written in the form

$$\theta(y) = 2SR_T y - \frac{1}{\Xi}\sum_{n=1}^{\infty} \frac{d^2 u_n(y)}{dy^2} \varepsilon^n. \tag{61}$$

Equations (36) and (61) yield the following expressions of Nu_1 and Nu_2:

$$Nu_1 = \frac{2SR_T + \sum_{n=1}^{\infty} a_n \varepsilon^n}{R_T\left[SR_T - 1 + \sum_{n=1}^{\infty} c_n \varepsilon^n\right] + 1} \tag{62}$$

$$Nu_2 = \frac{2SR_T + \sum_{n=1}^{\infty} b_n \varepsilon^n}{R_T\left[SR_T - 1 + \sum_{n=1}^{\infty} c_n \varepsilon^n\right] + 1} \tag{63}$$

where the coefficients a_n, b_n and c_n are given by

$$a_n = -\frac{1}{\Xi}\frac{\mathrm{d}^3 u_n}{\mathrm{d}y^3}\bigg|_{y=-1/4}, \quad b_n = -\frac{1}{\Xi}\frac{\mathrm{d}^3 u_n}{\mathrm{d}y^3}\bigg|_{y=1/4}$$

$$c_n = -\frac{1}{\Xi}\left(\frac{\mathrm{d}^2 u_n}{\mathrm{d}y^2}\bigg|_{y=1/4} - \frac{\mathrm{d}^2 u_n}{\mathrm{d}y^2}\bigg|_{y=-1/4}\right). \tag{64}$$

Equations (23), (52) and (57) yield the expression of the mean dimensionless velocity

$$\bar{u} = 1 + \sum_{n=1}^{\infty} d_n \varepsilon^n \tag{65}$$

where the coefficients d_n are given by

$$d_n = 2\int_{-1/4}^{1/4} u_n(y)\,\mathrm{d}y. \tag{66}$$

5. Analysis of the results

By means of the perturbation method described in the preceding section, both conditions of asymmetric fluid temperatures ($T_1 < T_2, R_T = 1$) and of symmetric fluid temperatures ($T_1 = T_2, R_T = 0$) have been analysed.

Let us first consider the condition $T_1 < T_2$, $R_T = 1$. In this case, u and θ depend on the dimensionless parameters ε, Ξ, Bi_1 and Bi_2. When the flow is upward, ε and Ξ are positive; when the flow is downward, ε and Ξ are negative.

Plots of u and θ vs. y for some values of ε and $Bi_1 = Bi_2 = 10$ are reported in Figs 4–6. In particular, Fig. 4 refers to $\Xi = 100$, Fig. 5 to $\Xi = 500$ and Fig. 6 to $\Xi = -300$. The number of terms of the perturbation series which is sufficient to attain convergence depends both on Ξ and on ε. The plots which appear in Fig. 4 have been obtained with 20 terms of the perturbation series, the plots in Fig. 5 with 30 terms and those in Fig. 6 with 25 terms. Figures 4 and 5 show that, for upward flow, both the dimensionless velocity and the dimensionless temperature, at each position, are increasing functions of ε. Moreover, the effect of ε on u is stronger for higher values of Ξ, while that on θ is weaker. Figure 6 shows that, for downward flow, at each position u is a decreasing function of $|\varepsilon|$, while θ is an increasing function of $|\varepsilon|$. The effect of ε on u and θ is stronger for upward flow than for downward flow. The results reported in Figs 4–6 can be explained qualitatively as follows. A stronger viscous dissipation causes higher fluid temperatures and, as a consequence, higher values of the buoyancy force. The increase of the buoyancy-force values yields an increase of the fluid velocity when the flow is upward and a decrease of the fluid velocity when the flow is downward.

In Fig. 7, Nu_1 and Nu_2 are plotted vs. $|\varepsilon|$ for $Bi_1 = Bi_2 = 10$ and for three values of Ξ. Figure 7 reveals that Nu_1 is an increasing function of $|\varepsilon|$, while Nu_2 is a decreasing function of $|\varepsilon|$. Moreover, it shows that, for

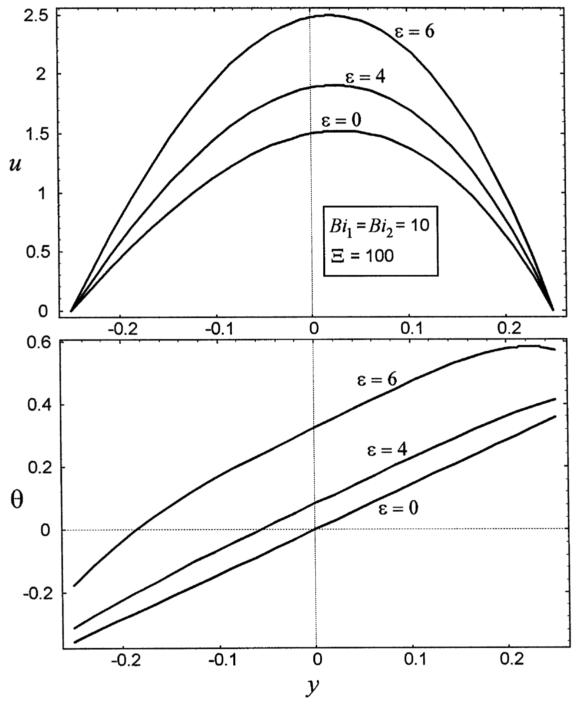

Fig. 4. Plots of u and θ vs. y in the case $R_T = 1$, for some values of ε, $\Xi = 100$ and $Bi_1 = Bi_2 = 10$.

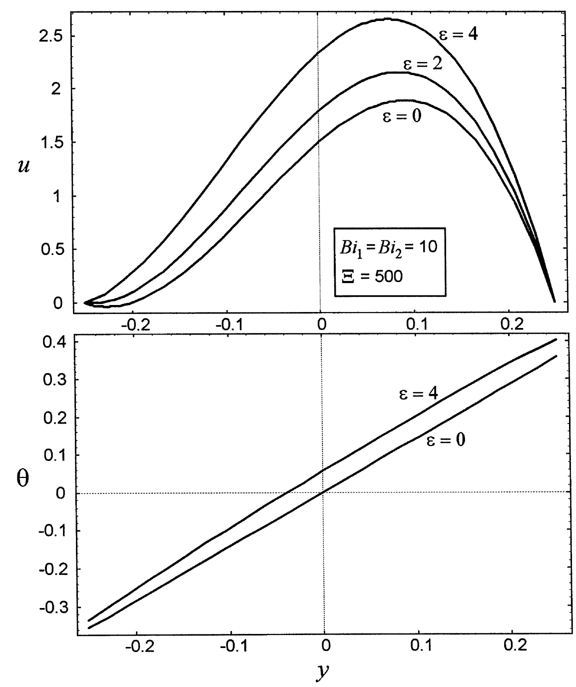

Fig. 5. Plots of u and θ vs. y in the case $R_T = 1$, for some values of ε, $\Xi = 500$ and $Bi_1 = Bi_2 = 10$.

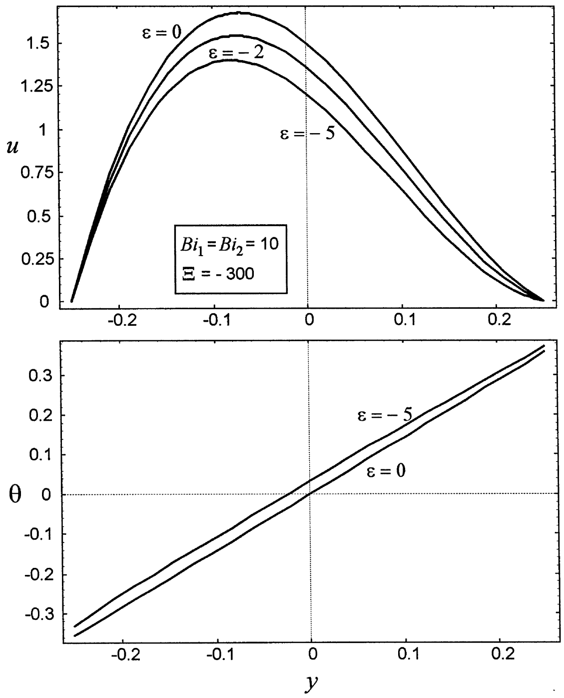

Fig. 6. Plots of u and θ vs. y in the case $R_T = 1$, for some values of ε, $\Xi = -300$ and $Bi_1 = Bi_2 = 10$.

upward flow, the effect of ε on Nu_1 and on Nu_2 is stronger for lower values of Ξ. In Fig. 8, $\bar{u}$ is plotted vs. $|\varepsilon|$ for $Bi_1 = Bi_2 = 10$ and for three values of Ξ. As one expects, $\bar{u}$ is an increasing function of $|\varepsilon|$ when the flow is upward, while it is a decreasing function of $|\varepsilon|$ when the flow is downward. For upward flow, the effect of ε on $\bar{u}$ is stronger for higher values of Ξ.

In Fig. 9, plots of u and θ vs. y with $Bi_1 = 0.1$ and $Bi_2 = 10$ are reported for some values of ε. The plots

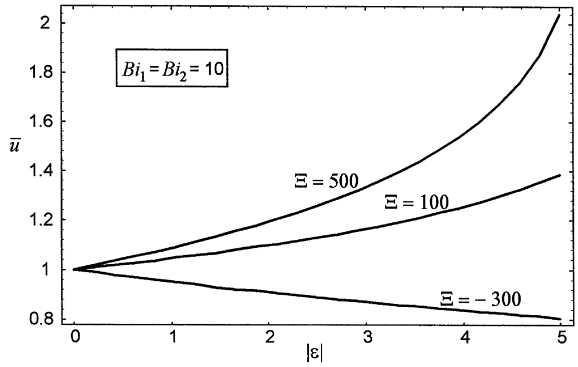

Fig. 8. Plots of $\bar{u}$ vs. the absolute value of ε in the case $R_T = 1$ for some values of Ξ and $Bi_1 = Bi_2 = 10$.

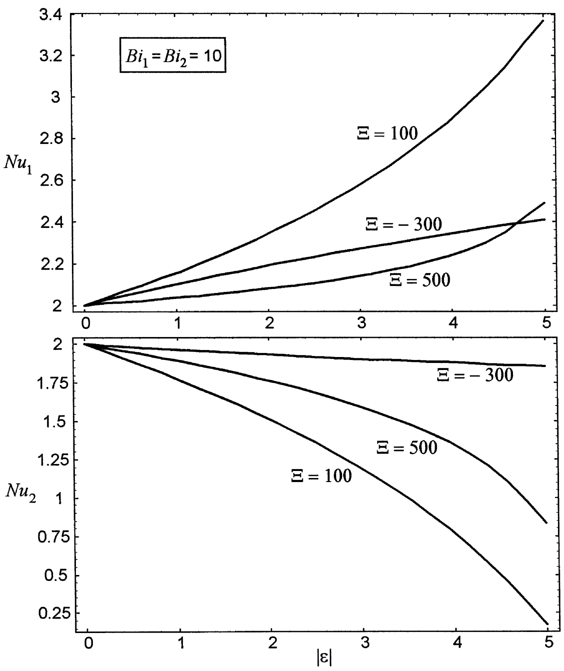

Fig. 7. Plots of Nu_1 and Nu_2 vs. the absolute value of ε in the case $R_T = 1$, for some values of Ξ and $Bi_1 = Bi_2 = 10$.

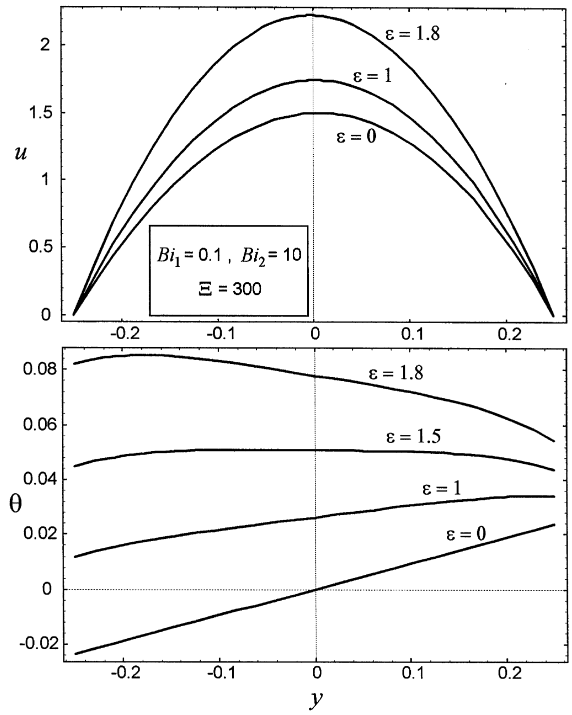

Fig. 9. Plots of u and θ vs. y in the case $R_T = 1$, for some values of ε, $\Xi = 300$, $Bi_1 = 0.1$ and $Bi_2 = 10$.

which appear in Fig. 9 refer to $\Xi = 300$ and are obtained with 27 terms of the perturbation series. Figure 9 shows that, when ε increases, θ increases more at $y = -1/4$ than at $y = 1/4$, i.e., θ increases more at the wall which has the smaller external-convection coefficient. In particular, for $\varepsilon = 1.5$ and $\varepsilon = 1.8$, the temperature at $y = -1/4$ exceeds that at $y = 1/4$, although $T_1 < T_2$. A comparison between Fig. 9 and Figs 6–7 reveals that the effect of ε on u and on θ, for a fixed value of Ξ, becomes stronger if either Bi_1 or Bi_2 becomes smaller.

Let us now consider the condition $T_1 = T_2$, $R_T = 0$. Equations (25)–(28) show that the dimensionless velocity u is a function of y which depends only on the dimensionless parameters ε, Bi_1 and Bi_2. Therefore, on account of equation (30), also $\Xi\theta$ is a function of y which depends only on ε, Bi_1 and Bi_2. Moreover, equation (36) ensures that $\Xi\,Nu_1$ and $\Xi\,Nu_2$ are uniquely determined by ε, Bi_1 and Bi_2. As in the case of asymmetric fluid temperatures, both ε and Ξ are positive when the flow is upward, while they are negative when the flow is downward.

In Figs 10 and 11, the dimensionless velocity u and the product $\Xi\theta$ are plotted vs. y for some values of ε. Figure 10 refers to $Bi_1 = Bi_2 = 10$, while Fig. 11 refers to $Bi_1 = 0.1$ and $Bi_2 = 10$. Figures 10 and 11 show that, at any given position, both u and $\Xi\theta$ are increasing functions of ε. As in the case of asymmetric fluid temperatures, the effect of viscous dissipation on the dimensionless velocity profile and on the dimensionless temperature profile is more significant in the case of upward flow ($\varepsilon > 0$) than in the case of downward flow ($\varepsilon < 0$), and becomes stronger if either Bi_1 or Bi_2 becomes smaller.

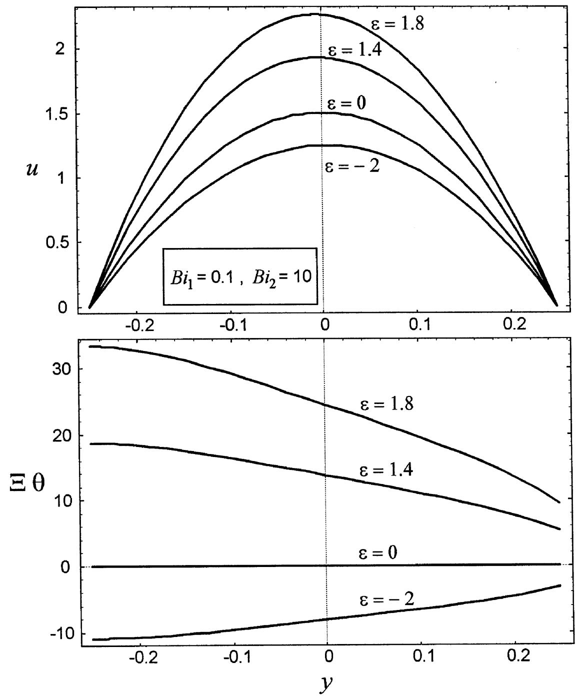

Fig. 11. Plots of u and $\Xi\theta$ vs. y in the case $R_T = 0$, for some values of ε, $Bi_1 = 0.1$ and $Bi_2 = 10$.

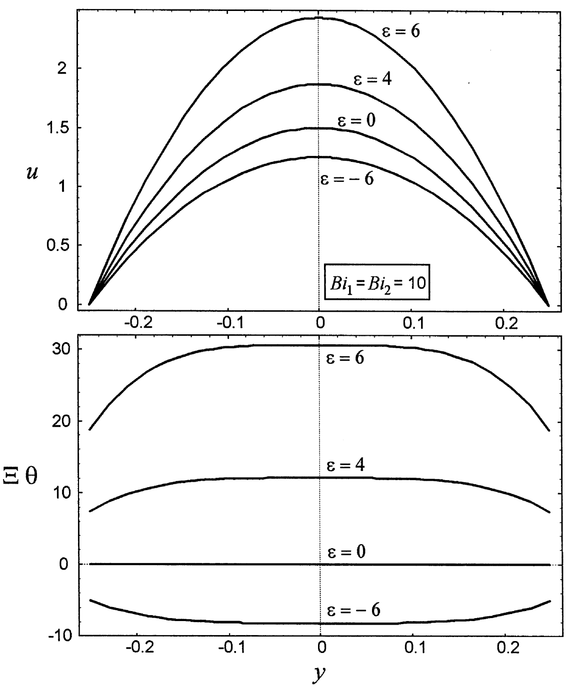

Fig. 10. Plots of u and $\Xi\theta$ vs. y in the case $R_T = 0$, for some values of ε and $Bi_1 = Bi_2 = 10$.

Clearly, when $Bi_1 = Bi_2$ the dimensionless temperature profile is symmetric, so that, on account of equation (36), $Nu_1 = Nu_2 = Nu$. Values of $\Xi\,Nu$ for this condition are reported in Table 1 for several values of ε and Bi. The table shows that $\Xi\,Nu$ is an increasing function of ε for every value of Bi, and that the effect of ε on $\Xi\,Nu$ becomes stronger when the Biot number becomes lower. Finally, the effect of ε on $\Xi\,Nu$ is more important for upward flow than for downward flow. The values of $\Xi\,Nu$ reported in Table 1 have been obtained with 30 terms of the perturbation series for $Bi \geqslant 20$, with 35 terms for $Bi = 10$, and with 46 terms for $Bi = 7$. In each case, evaluations with 27, 32 and 43 perturbation terms, respectively for $Bi \geqslant 20$, $Bi = 10$ and $Bi = 7$, have given the same results as those reported in Table 1, at least for the digits which appear in the table. The results obtained for $Bi = 10^5$ are in perfect agreement with those reported in ref. [11] for the boundary condition of prescribed wall temperatures.

6. Conclusions

The laminar and fully developed mixed convection with viscous dissipation in a plane vertical channel has

Table 1
Values of $\Xi\,Nu$ as a function of ε and Bi, for $R_T = 0$ and $Bi_1 = Bi_2$ (completely symmetric case)

ε	$Bi = 10^5$ $\Xi\,Nu$	$Bi = 50$ $\Xi\,Nu$	$Bi = 20$ $\Xi\,Nu$	$Bi = 10$ $\Xi\,Nu$	$Bi = 7$ $\Xi\,Nu$
−4.0	−43.211	−41.790	−39.858	−37.072	−35.027
−3.5	−38.275	−37.149	−35.603	−33.343	−31.660
−3.0	−33.219	−32.362	−31.173	−29.409	−28.077
−2.5	−28.036	−27.420	−26.554	−25.250	−24.249
−2.0	−22.722	−22.313	−21.731	−20.840	−20.143
−1.5	−17.269	−17.030	−16.685	−16.148	−15.720
−1.0	−11.669	−11.559	−11.398	−11.141	−10.932
−0.5	−5.916	−5.887	−5.845	−5.775	−5.718
0.0	0.000	0.000	0.000	0.000	0.000
0.5	6.087	6.119	6.166	6.248	6.320
1.0	12.356	12.487	12.690	13.048	13.375
1.5	18.817	19.125	19.612	20.499	21.346
2.0	25.482	26.056	26.983	28.733	30.491
2.5	32.365	33.306	34.860	37.923	41.196
3.0	39.478	40.904	43.315	48.309	54.078
3.5	46.839	48.885	52.437	60.238	70.231
4.0	54.464	57.286	62.335	74.234	91.906

been analysed. The boundary condition of convective heat exchange with an external fluid at each boundary plane has been considered. The simpler cases of either negligible viscous dissipation or negligible buoyancy forces have been solved analytically. The combined effects of buoyancy forces and viscous dissipation have been studied by a perturbation series method. The pure number $\varepsilon = Br\,Gr/Re$ has been chosen as the perturbation parameter. Both the case of asymmetric fluid temperatures ($R_T = 1$), with either equal or different Biot numbers, and the case of symmetric fluid temperatures ($R_T = 0$), with either equal or different Biot numbers, have been considered. The results can be summarized as follows. For upward flow, both the dimensionless velocity u and the dimensionless temperature θ, at each position, are increasing function of ε, i.e., of the viscous-dissipation parameter. The effect of ε on u, on θ and on the Nusselt numbers increases when at least one of the Biot number decreases. For downward flow, at each position, u is a decreasing function of $|\varepsilon|$ while θ is an increasing function of $|\varepsilon|$. The effect of ε on u, on θ and on the Nusselt numbers is more relevant for upward flow than for downward flow. In the completely symmetric case ($R_T = 0, Bi_1 = Bi_2$) the value of $\Xi\,Nu$ is uniquely determined by ε and Bi. A table of $\Xi\,Nu$ as a function of ε and Bi has been reported. The table shows that, for each value of Bi, $\Xi\,Nu$ is an increasing function of ε, both for downward and for upward flow. The effect of ε on $\Xi\,Nu$ for upward flow is more relevant than that for downward flow, and becomes stronger when the Biot number decreases.

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